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A note on supplementary difference sets

Abstract

Let S_1, S_2, \dots, S_n be subsets of G , a finite abelian group of order v , containing k_1, k_2, \dots, k_n elements respectively. Write T_i for the totality of all differences between elements of S_i (with repetitions), and T for the totality of elements of all the T_i . We will denote this by $T = T_1 \cup T_2 \cup \dots \cup T_n$. If T contains each non-zero element of G a fixed number of times, λ say, then the sets S_1, S_2, \dots, S_n will be called n - $\{v; k_1, k_2, \dots, k_n; \lambda\}$ supplementary difference sets.

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A Note on Supplementary Difference Sets

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Let S_1, S_2, \dots, S_n be subsets of G , a finite abelian group of order v , containing k_1, k_2, \dots, k_n elements respectively. Write T_i for the totality of all differences between elements of S_i (with repetitions), and T for the totality of elements of all the T_i . We will denote this by $T = T_1 \& T_2 \& \dots \& T_n$. If T contains each non-zero element of G a fixed number of times, λ say, then the sets S_1, S_2, \dots, S_n will be called $n - \{v; k_1, k_2, \dots, k_n; \lambda\}$ *supplementary difference sets*.

If $k_1 = k_2 = \dots = k_n = k$ we will write $n - \{v; k; \lambda\}$ to denote the supplementary difference sets. If $k_1 = k_2 = \dots = k_i, k_{i+1} = k_{i+2} = \dots = k_{i+j}, \dots, k_l = \dots = k_n$ then sometimes we write $n - \{v; i: k_1, j: k_{i+1}, \dots; \lambda\}$. It can be easily seen by counting the differences that the parameters of $n - \{v; k_1, k_2, \dots, k_n; \lambda\}$ supplementary difference sets satisfy

$$\lambda(v-1) = \sum_{j=1}^n k_j(k_j-1).$$

We use braces, $\{ \}$, to denote sets and square brackets, $[\]$, to denote collections where repetitions may remain.

We now let $v = 4r(2\lambda+1)+1 = p^y$, where p is a prime and further let

$$H_i = \{x^{4rj+i}: 0 \leq j \leq 2\lambda\}, \quad i = 0, 1, \dots, 4r-1$$

with x a primitive element of $GF(v)$. Write

$$L = H_{2i_1} \cup H_{2i_2} \cup \dots \cup H_{2i_m}$$

for some $m, 0 < m < 2r$, where the i_j are distinct integers. Now we consider the differences between elements of H_{2i} , that is, the collection

$$\begin{aligned} & [x^{4rj+2i} - x^{4rl+2i}: j \neq l, 0 \leq j, l \leq 2\lambda] \\ &= \{x^{4rj+2i}: 0 \leq j \leq 2\lambda\} \text{ times } [1 - x^{4r(l-j)}: l \neq j, 0 \leq l \leq 2\lambda] \\ &= H_{2i} \text{ times } [1 - x^{4r(l-j)}: l \neq j, 0 \leq l \leq 2\lambda] \end{aligned} \quad (1)$$

and, since any element of a group multiplied onto a coset gives a coset, this expression must represent cosets with certain multiplicities, say b_k , write

$$= \sum_{k=0}^{4r-1} b_k H_k, \quad (2)$$

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where, since H_{2i} has $2\lambda + 1$ elements, the number of elements in (1) is $2\lambda(2\lambda + 1)$ and the number of elements in (2) is $\sum_{k=0}^{4r-1} b_k(2\lambda + 1)$. So

$$\sum_{k=0}^{4r-1} b_k = 2\lambda.$$

Now $2\lambda + 1$ is odd, so $-1 \in H_{2r}$. Then if $x^a - x^b$ appears in (1) so does $x^b - x^a$. Thus whenever an element y occurs so does $-y$ and $y \in H_c \Rightarrow -y \in H_{c+2r}$. Thus $b_k = b_{k+2r}$.

The differences between elements of H_{2i} and H_{2k} are given by the collection

$$[x^{4rj+2i} - x^{4rl+2k}; 0 \leq j, l \leq 2\lambda] \quad (3)$$

$$= \{x^{4rj+2i}; 0 \leq j \leq 2\lambda\} \text{ times } [1 - x^{4r(l-j)+2(k-i)}; 0 \leq l \leq 2\lambda]$$

$$= H_{2i} \text{ times } [1 - x^{4r(l-j)+2(k-i)}; 0 \leq l \leq 2\lambda]$$

$$= \sum_{n=0}^{4r-1} c_n H_n \quad (4)$$

where c_n give the multiplicities. By the same reasoning as before,

$$\sum_{n=0}^{4r-1} c_n = 2\lambda + 1.$$

Now consider the differences from L , that is

$$[\text{differences from } H_{2i_j}; j = 1, 2, \dots, m] \quad (5)$$

$$\& [\text{differences from } H_{2i_j} - H_{2i_k}; i_j \neq i_k, 0 \leq i_j, i_k \leq m] \quad (6)$$

$$= \sum_{k=0}^{4r-1} a_k H_k \quad \text{using (2) and (4)}. \quad (7)$$

Counting elements we see (5) and (6) have $m(2\lambda + 1)$ and $(m(2\lambda + 1) - 1)$ and (7) has $(2\lambda + 1) \sum_{k=0}^{4r-1} a_k$ elements. Hence

$$\sum_{k=0}^{4r-1} a_k = m(m(2\lambda + 1) - 1).$$

Finally, we note that in (6) if $H_a - H_b$ occurs so does $H_b - H_a$ so if y occurs so does $-y$ and as before we see that

$$a_k = a_{k+2r}. \quad (8)$$

Write

$$\left. \begin{aligned} w &= \sum_{k=0}^{r-1} a_{2k} - \sum_{k=0}^{r-1} a_{2k+1} \\ z &= (w, w + m), \quad s = |w + m|/z, \quad t = |w|/z. \end{aligned} \right\} \quad (9)$$

We now show, using L to construct sets of size $m(2\lambda+1)$ and $m(2\lambda+1)+1$, how to find some supplementary difference sets.

THEOREM 1. *Let $v=4r(2\lambda+1)+1=p^\gamma$, where p is a prime and $r=2^\delta$. Then s copies of each of*

$$L_j = x^{2^j}L, \quad j = 0, 1, \dots, r-1,$$

and t copies of each of

$$K_j = 0 \cup x^{2^{j+i}}L, \quad j = 0, 1, \dots, r-1,$$

where s, t and w are given by (9), $i=0$ if (w is negative and $m > -w$), $i=1$ otherwise, are

$$r(s+t) - \{4r(2\lambda+1)+1; rt; m(2\lambda+1)+1; rs; m(2\lambda+1);$$

$$\varphi_{\frac{1}{4}}[m^2(2\lambda+1)(t+s)+m(t-s)]\}$$

supplementary difference sets.

Proof. Since $2\lambda+1$ is always odd, $-1 \in H_{2r}$, we have from (8) $a_k = a_{k+2r}$. The totality of differences from

$$L_j = H_{2i_1+2j} \cup H_{2i_2+2j} \cup \dots \cup H_{2i_m+2j}$$

is x^{2^j} times the totality of differences from L_0 or

$$\&_{k=0}^{4r-1} a_{4r-2j+k} H_k = \&_{k=0}^{2r-1} a_{2r-2j+k} (H_k \& H_{k+2r}).$$

So by taking all the differences from $L_j, j=0, 1, \dots, r-1$ we have

$$\begin{aligned} X &= \&_{i=0}^{2r-1} \left\{ \left(\sum_{k=0}^{r-1} a_{2k} \right) H_{2i} \& \left(\sum_{k=0}^{r-1} a_{2k+1} \right) H_{2i+1} \right\} \\ &= \&_{i=0}^{2r-1} (\alpha H_{2i} \& \beta H_{2i+1}). \end{aligned}$$

The totality of differences, then, from the sets

$$K_j = 0 \cup H_{2i_1+2j+1} \cup H_{2i_2+2j+1} \cup \dots \cup H_{2i_m+2j+1}, \quad j = 0, 1, \dots, r-1,$$

$$\text{is } Z = \&_{i=0}^{2r-1} (\beta H_{2i} \& (\alpha+m) H_{2i+1}).$$

There are four cases to consider:

- (i) $\alpha \geq \beta$ and $\beta \geq \alpha+m$, which is impossible;
- (ii) $\alpha \leq \beta$ and $\beta \leq \alpha+m$. Here $w = \alpha - \beta$ is negative and $m > \beta - \alpha = -w$.

So, if instead of the sets K_j we use the totality of differences from the sets $0 \cup L_j$, then we have the differences

$$Y = \bigcup_{i=0}^{2r-1} ((\alpha + m) H_{2i} \cup \beta H_{2i+1a}).$$

Now s times X plus t times Y (where s and t are defined in (9)) gives $(\beta m/z) G$;

(iii) $\alpha < \beta$ and $\beta \geq \alpha + m$; and

(iv) $\alpha > \beta$ and $\beta \leq \alpha + m$.

In these last two cases s times X and t times Z gives

$$((\beta^2 - \alpha^2 - \alpha m)/z) G \quad \text{and} \quad ((\alpha^2 + \alpha m - \beta^2)/z) G$$

respectively.

Then, noting that by summing the elements of X in two ways we find $\alpha + \beta = \frac{1}{2}m[m(2\lambda + 1) - 1]$, we have the result of the theorem.

EXAMPLE. With $v=41$, $r=2$, $\lambda=2$, and $m=3$, $w=1$, $s=2$, $t=1$ we find $6 - \{41; 2:16, 4:15; 33\}$ supplementary difference sets.

In the theorem the initial set L has been left reasonable undecided but if we choose another initial set.

$$M_j = H_{2j_1+2j} \cup H_{2j_2+2j} \cup \dots \cup H_{2j_m+2j} \quad j = 0, 1, \dots, r-1$$

where all the j_a are distinct, we may get a different set of supplementary difference sets.

For example: with $v=41$, $r=2$, $\lambda=2$, with $m=2$ and the initial set $H_0 \cup H_2$ we get $w=1$, $s=3$, $t=1$ and hence $8 - \{41; 2:11, 6:10; 19\}$ supplementary difference sets, while with the initial set $H_0 \cup H_4$ we get $w=-3$, $s=1$, $t=3$ and hence $8 - \{41; 6:11, 2:10; 21\}$ supplementary difference sets.

Finally we note that *balanced incomplete block designs* may be obtained from supplementary difference sets with two k values by using the results of Jennifer Wallis [2].

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